VOLUME-12 Part B and C

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IV. Quantum Mechanics

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13.1 Elementary theory of Scattering, Phase shifts and Partial Waves

Introduction

1. Classical Scattering Theory

Imagine a particle incident on some scattering center (say, a proton fired at a heavy nucleus). It comes in with energy E and impact parameter b, and it emerges at some scattering angle θ - see Figure 1. (I'll assume for simplicity that the target is azimuthally symmetrical, so the trajectory remains in one phase, and that the target is very heavy, so the recoil is negligible) The essential problem of classical scattering theory is this: Given the impact parameter, calculate the scattering angle. Ordinarily, of course the smaller the impact parameter, the greater the scattering angle.

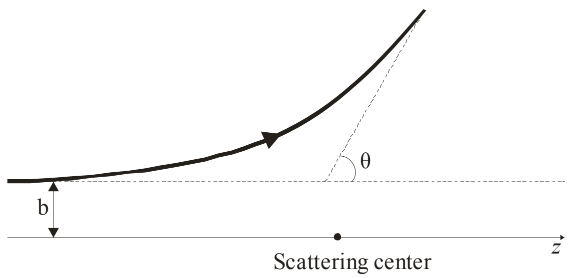

Figure.1 : The classical scattering problem, showing the impact parameter b and the scattering angle θ.

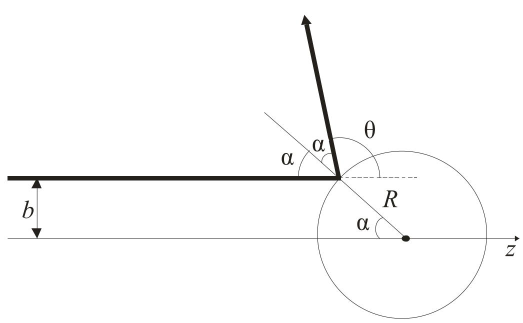

Figure. 2 : Elastic hard - sphere scattering.

More generally, particles incident within an infinitesimal patch of cross-sectional area $d\sigma$ will scatter into a corresponding infinitesimal solid angle $d\Omega$. The larger $d\sigma$ is, the bigger $d\Omega$ will be; the proportionality factor, $D(\theta) \equiv d\sigma / d\Omega$, is called the differential (scattering) cross-section:

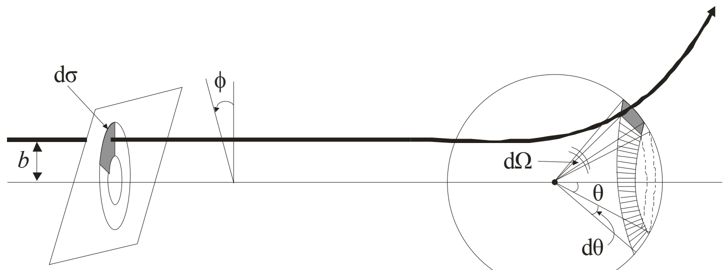

Figure. 3 : Particles incident in the area $d\sigma$ scater into the solid angle $d\Omega$.

$$d\sigma = D(\theta)d\Omega \qquad \ldots 1$$

In terms of the impact parameter and the azimuthal angle $\phi, d\sigma = b\, db\, d\phi$ and $d\Omega = \sin\theta\, d\theta\, d\phi,$ so

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|. \qquad \ldots 2$$

(Since θ is typically a decreasing function of b, the derivative is actually negative – hence the absolute value sign.)

The total cross-section is the integral of $D(\theta)$, over all solid angles:

$$\sigma \equiv \int D(\theta)d\Omega; \qquad \ldots 3$$

roughly speaking, it is the total area of incident beam that is scattered by the target. For example, in the case of hard-sphere scattering,

$$\sigma = \left(R^2/4\right)\int d\Omega = \pi R^2, \qquad \ldots 4$$

which is just what we would expect: It's the cross-sectional area of the sphere; BB's incident within this area will hit the target and those farther out will miss it completely. But the virtue of the formalism developed here is that it applies just as well to "soft" targets (such as the Coulomb field of a nucleus) that are not simply "hit-or-miss."

Finally, suppose we have a beam of incident particles, with uniform intensity (or luminosity, as particle physicists call it)

$$L \equiv \text{number of incident particles per unit area, per unit time.} \qquad \ldots 5$$

The number of particles entering area $d\sigma$ (and hence scattering into solid angle $d\Omega$), per unit time, is $dN = L\,d\sigma = L\; D(\theta)d\Omega,$ so

$$D(\theta) = \frac{1}{L}\frac{dN}{d\Omega} \qquad \ldots 6$$

This is often taken as the definition of the differential cross-section, because it makes reference only to quantities easily measured in the laboratory: If the detector accepts particles scattering into a solid angle $d\Omega$, we simply count the number recorded, per unit time, divide by $d\Omega$, and normalize to the luminosity of the incident beam.

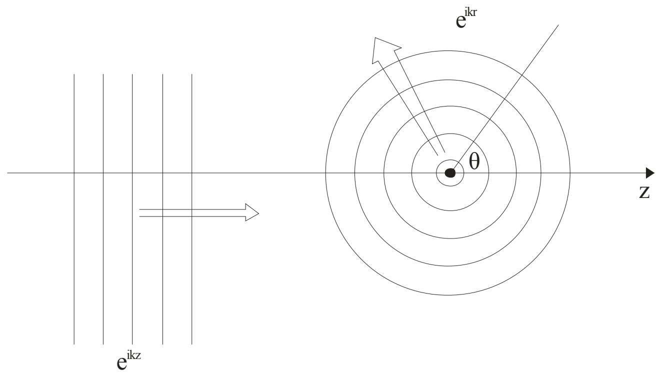

Figure. 4 : Scattering of waves; incoming plane wave generates outgoing spherical wave.

2. Quantum Scattering Theory

In the quantum theory of scattering, we imagine an incident plane wave, $\psi(z) = Ae^{ikz}$, travelling in the z direction, which encounters a scattering potential, producing an outgoing spherical wave. That is, we look for solutions to the Schrodinger equation of the general form

$$\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}, \text{for large } r. \qquad \ldots 7$$

(The spherical wave carries a factor of 1/*r*, because this portion of $|\psi|^2$ must go like $1/r^2$ to conserve probability.) The wave number *k* is related to the energy of the incident particles in the usual way:

$$k \equiv \frac{\sqrt{2mE}}{\hbar} \qquad \ldots 8$$

As before, I shall assume the target is azimuthally symmetrical; in the more general case the amplitude *f* of the outgoing spherical wave could depend on ϕ as well as θ.

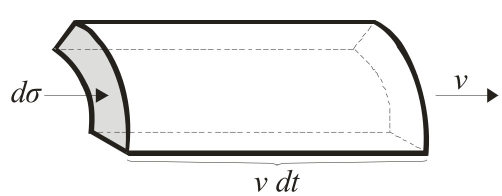

Fig. 5 : The volume dV of incident beam that passes through area $d\sigma$ in time dt.

The whole problem is to determine the scattering amplitude $f(\theta)$; it tells you the probability of scattering in a given direction θ, and hence is related to the differential cross-section. Indeed, the probability that the incident particle, travelling at speed υ, passes through the infinitesimal area $d\sigma$, in time dt, is

$$dP = |\psi_{incident}|^2 dV = |A|^2 (\upsilon dt) d\sigma$$

But this is equal to the probability that the particle scatters into the corresponding solid angle $d\Omega$:

$$dP = |\psi_{scattered}|^2 dV = \frac{|A|^2 |f|^2}{r^2} (\upsilon dt) r^2 d\Omega$$

from which it follows that $d\sigma = |f^2| d\Omega$, and hence

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \qquad \ldots 9$$

Evidently the differential cross-section (which is the quantity of interest to the experimentalist) is equal to the absolute square of the scattering amplitude (which is obtained by solving the Schrodinger equation). In the following sections we will study two techniques for calculating the scattering amplitude: partial wave analysis and the Born approximation.

13.2 Phase Shifts

Consider first the problem of one-dimensional scattering from a localized potential $V(x)$ on the half-line $x < 0$. I'll put a "brick wall" at $x = 0$, so a wave incident from the left,

$$\psi_i(x) = Ae^{ikx} \quad (x < -a) \qquad \ldots 1$$

is entirely reflected

$$\psi_r(x) = Be^{-ikx} \quad (x < -a) \qquad \ldots 2$$

Whatever happens in the interaction region $(-a < x < 0)$, the amplitude of the reflected wave has got to be the same as that of the incident wave, by conservation of probability. But it need not have the same phase. If there were no potential at all (just the wall at $x = 0$), then $B = -A$, since the total wave function (incident plus reflected) must vanish at the origin:

$$\psi_0(x) = A\left(e^{ikx} - e^{-ikx}\right) \quad (V(x) = 0) \qquad \ldots 3$$

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