VOLUME-11 Part B and C

CONTENTS

IV. Quantum Mechanics

IV.11. Angular Momentum algebra and Spin

11.1. Angular Momentum

Classically, the angular momentum of a particle is given by,

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix}$$

$$\vec{L} = \hat{i}\left(yP_z - zP_y\right) - \hat{j}\left(xP_z - zP_x\right) + \hat{k}\left(xP_y - yP_x\right)$$

$$\vec{L} = \hat{i}\left(yP_z - zP_y\right) + \left(zP_x - xP_z\right)\hat{j} + \left(xP_y - yP_x\right)$$

$$L_x = yP_x - zP_y$$

$$L_y = zP_x - xP_z$$

$$L_z = xP_y - yP_x$$

Also, $$P_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$P_y = \frac{\hbar}{i}\frac{\partial}{\partial y}$$

$$P_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$$

We will obtain the eigenvalues of angular momentum operator by algebraic technique.

Eigenvalues:

The operators *Lx* and *Ly* do not commute.

1. $[L_x, L_y] = [yP_z - zP_y, zP_x - xP_z]$

$$= \left[yP_z, zP_x\right] - \left[zP_y, zP_x\right] - \left[yP_z, xP_z\right] + \left[yP_y, xP_z\right]$$

$$= [yP_z, zP_x] + [zP_y, xP_z] - 0$$

Since, $[r_i, r_i] = 0$ and $[P_i, P_i] = [P_j, P_j] = [P_k, P_k] = 0$

$\therefore$ $$[L_x, L_y] = yP_x[P_z, Z] + xP_y[Z, P_z]$$

Since $[x, P] = i\hbar$ and $[P, x] = -i\hbar$

$\therefore \quad [L_x, L_y] = yP_x(-i\hbar) + xP_y\ (i\hbar)$

$= i\hbar\left(xP_y - yP_z\right)$

$[L_x, L_y] = i\hbar L_z$

Similarly,

2. $[L_y, L_z] = [zP_x - xP_z, xP_y - yP_x]$

$= [, zP_x\ xP_y] - [xP_z,\ xP_y] - [zP_x, yP_x] + [xP_z, yP_x]$

$= [zP_x, xP_y] - 0 - 0 + [xP_z, yP_x]$

$= zP_y[P_x,\ x] + yP_z[x, P_x]$

$\therefore \quad [L_y, L_z] = zP_y(-i\hbar) + yP_z(i\hbar)$

$= i\hbar(yP_z - zP_y)$

$[L_y, L_z] = i\hbar L_x$

3. $[L_z, L_x] = [xP_y - yP_x, yP_z + zP_y]$

$= [xP_y, yP_z] - [yP_x, yP_z] - [xP_y, zP_y] + [yP_x,\ zP_y]$

$= [xP_y,\ yP_z] - 0 - 0 + [yP_x,\ zP_y]$

$= xP_z[P_y,\ y] + zP_x[y,\ P_y]$

$= xP_z(-i\hbar) + zP_x(i\hbar)$

$= i\hbar(zP_x - xP_z)$

$[L_z, L_x] = i\hbar L_y$

Now,

(I) $[L_y, L_x] = [zP_x - xP_z, yP_z - zP_y]$

$= [zP_x,\ yP_z] - [xP_z,\ yP_z] - [zP_x, \text{zPy}] + [xP_z,\ yP_z]$

$= [zP_x,\ yP_z] - 0 - 0 + [xP_z,\ zP_y]$

$= yP_x[z,\ P_z] + xP_y[P_z, z]$

$= yP_x(i\hbar) + xP_y\ (-i\hbar)$

$\therefore \quad [L_y, L_x] = i\hbar\ (yP_x - xP_y)$

$= -i\hbar\ (xP_y - xP_y)$

$$[L_y, L_x] = -i\hbar L_z$$

(II) $[L_z, L_y] = [xP_y - yP_x, zP_x - xP_z]$

$$= [xP_y, zP_x] - [yP_x, zP_x] - [xP_y, xP_z] + [yP_x, xP_z]$$

$$= [xP_y,\ zP_x] - 0 - 0 + [yP_x, xP_z]$$

$$= zP_y[x,\ P_x] + yP_z[P_x,\ x]$$

$$= zP_y(i\hbar) + yP_z(-i\hbar)$$

$$= i\hbar\ (zP_y - yP_z)$$

$$= -i\hbar\ (yP_z - zP_y)$$

$$[L_z, L_y] = -i\hbar L_x$$

(III) $[L_x, L_z] = [yP_z - zP_y, xP_y - yP_x]$

$$= [yP_z, xP_y] - [zP_y,\ xP_y] - [yP_z,\ yP_x] + [zP_y,\ yP_x]$$

$$= [yP_z, xP_y] - 0 - 0 + [zP_y,\ yP_x]$$

$$= [yP_z, xP_y] + [zP_y,\ yP_x]$$

$$= xP_z[y\ ,\ P_y] + zP_x[P_y\ ,\ y]$$

$$\therefore \qquad [L_x, L_z] = xP_z(i\hbar) + zP_x(-i\hbar)$$

$$= i\hbar(xP_z - zP_x)$$

$$= -i\hbar\ (zP_x - xP_z)$$

$[L_x, L_z] = -i\hbar L_y$ Ly

These are the fundamental commutation relation for angular momentum.

L_x, L_y and L_z are incompatible observables.

According to the generalized uncertainty principle

$$\sigma_{Lx}^2 \sigma_{Ly}^2 \geq \left(\frac{1}{2i} \langle [L_x, L_y] \rangle \right) 2$$

$$\sigma_{Lx}^2 \sigma_{Ly}^2 \geq \left(\frac{1}{2i} \langle i\hbar\, L_z \rangle \right) 2$$

$$\sigma_{Lx}^2\sigma_{Ly}^2 \geq \left(\frac{\hbar}{2}\langle L_z\rangle\right)2$$

$$\sigma_{Lx}^2\sigma_{Ly}^2 \geq \frac{\hbar}{2}\left|\langle L_z\rangle\right|$$

The square of the total angular momentum is,

$$\mathrm{L}^2 = \mathrm{L}_x^2 + \mathrm{L}_y^2 + \mathrm{L}_z^2$$

(1) $$[L^2 , L_x] = [L_x^2 , L_x] + [L_y^2 , L_x] + [L_z^2 , L_x]$$

$$= 0 + L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$$

$$= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_y)L_z$$

$$= -i\hbar\, L_yL_z - i\hbar\, L_yL_z + i\hbar\, L_yL_z + i\hbar\, L_yL_z$$

$$[L^2, L_x] = 0$$

Similarly, $$[L^2, L_y] = 0$$

$$[L^2, L_z] = 0$$

$\therefore$ $$\left[L^2, \vec{L}\right] = 0$$

$\therefore$ L^2 is compatible with each component of $\vec{L}$. To find eigenstates of L^2 and L_z.

$$L^2 f = Af \quad \text{and} \quad Lzf = Mf$$

Now, Using "ladder operator" technique.

Let,

$$L_\pm = L_x \pm iL_y$$

Now

$$[L_z, L_\pm] = [L_z, L_x] \pm i\,[L_z, L_y]$$

$$= (i\,\hbar\, L_y) \pm i(-i\,\hbar\, L_x)$$

$$= \pm i\,\hbar\, L_y \pm \hbar\, L_x$$

$$= \hbar\,[\pm L_x + i\, L_y]$$

$$= \pm\hbar\,[L_x \pm i\, L_y]$$

Continued with...Page 5 Onwards.... It's So Goooood!!!, Buy it now...!