

**VOLUME-11**

**Part B and C**

**CONTENTS**

**IV. Quantum Mechanics**

8.1 Angular Momentum algebra	1
8.2 Spin	9
8.3 Additional of angular momenta	23
9.0 Hydrogen Atom	29
10.1 Time-independent perturbation Theory and applications	56
10.2 The fine Structure of Hydrogen	65
10.3 Spin-orbit coupling	68
10.4 The Zeeman effect	72
11.1 The Variational principle	80
11.2 The WKB approximation	90
12.1 Time Dependent Perturbation Theory	97

## IV.11. Angular Momentum algebra and Spin

### 11.1. Angular Momentum

Classically, the angular momentum of a particle is given by,

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix}$$

$$\vec{L} = \hat{i}(yP_z - zP_y) - \hat{j}(xP_z - zP_x) + \hat{k}(xP_y - yP_x)$$

$$\vec{L} = \hat{i}(yP_z - zP_y) + (zP_x - xP_z)\hat{j} + (xP_y - yP_x)\hat{k}$$

$$L_x = yP_z - zP_y$$

$$L_y = zP_x - xP_z$$

$$L_z = xP_y - yP_x$$

Also, 
$$P_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

$$P_y = \frac{\hbar}{i} \frac{\partial}{\partial y}$$

$$P_z = \frac{\hbar}{i} \frac{\partial}{\partial z}$$

We will obtain the eigenvalues of angular momentum operator by algebraic technique.

#### Eigenvalues:

The operators  $L_x$  and  $L_y$  do not commute.

$$\begin{aligned} 1. [L_x, L_y] &= [yP_z - zP_y, zP_x - xP_z] \\ &= [yP_z, zP_x] - [zP_y, zP_x] - [yP_z, xP_z] + [yP_y, xP_z] \\ &= [yP_z, zP_x] + [zP_y, xP_z] - 0 \end{aligned}$$

Since,  $[r_i, r_i] = 0$  and  $[P_i, P_i] = [P_j, P_j] = [P_k, P_k] = 0$

$$\therefore [L_x, L_y] = yP_x [P_z, Z] + xP_y [Z, P_z]$$

Since  $[x, P] = i\hbar$  and  $[P, x] = -i\hbar$

$$\begin{aligned}\therefore [L_x, L_y] &= yP_x(-i\hbar) + xP_y(i\hbar) \\ &= i\hbar(xP_y - yP_x)\end{aligned}$$

$$[L_x, L_y] = i\hbar L_z$$

Similarly,

$$\begin{aligned}2. [L_y, L_z] &= [zP_x - xP_z, xP_y - yP_x] \\ &= [zP_x, xP_y] - [xP_z, xP_y] - [zP_x, yP_x] + [xP_z, yP_x] \\ &= [zP_x, xP_y] - 0 - 0 + [xP_z, yP_x] \\ &= zP_y[P_x, x] + yP_z[x, P_x]\end{aligned}$$

$$\begin{aligned}\therefore [L_y, L_z] &= zP_y(-i\hbar) + yP_z(i\hbar) \\ &= i\hbar(yP_z - zP_y)\end{aligned}$$

$$[L_y, L_z] = i\hbar L_x$$

$$\begin{aligned}3. [L_z, L_x] &= [xP_y - yP_x, yP_z + zP_y] \\ &= [xP_y, yP_z] - [yP_x, yP_z] - [xP_y, zP_y] + [yP_x, zP_y] \\ &= [xP_y, yP_z] - 0 - 0 + [yP_x, zP_y] \\ &= xP_z[P_y, y] + zP_x[y, P_y] \\ &= xP_z(-i\hbar) + zP_x(i\hbar) \\ &= i\hbar(zP_x - xP_z)\end{aligned}$$

$$[L_z, L_x] = i\hbar L_y$$

Now,

$$\begin{aligned}(I) [L_y, L_x] &= [zP_x - xP_z, yP_z - zP_y] \\ &= [zP_x, yP_z] - [xP_z, yP_z] - [zP_x, zP_y] + [xP_z, yP_z] \\ &= [zP_x, yP_z] - 0 - 0 + [xP_z, zP_y] \\ &= yP_x[z, P_z] + xP_y[P_z, z] \\ &= yP_x(i\hbar) + xP_y(-i\hbar)\end{aligned}$$

$$\begin{aligned}\therefore [L_y, L_x] &= i\hbar(yP_x - xP_y) \\ &= -i\hbar(xP_y - yP_x)\end{aligned}$$

$$[L_y, L_x] = -i\hbar L_z$$

$$\begin{aligned} \text{(II)} [L_z, L_y] &= [xP_y - yP_x, zP_x - xP_z] \\ &= [xP_y, zP_x] - [yP_x, zP_x] - [xP_y, xP_z] + [yP_x, xP_z] \\ &= [xP_y, zP_x] - 0 - 0 + [yP_x, xP_z] \\ &= zP_y [x, P_x] + yP_z [P_x, x] \\ &= zP_y (i\hbar) + yP_z (-i\hbar) \\ &= i\hbar (zP_y - yP_z) \\ &= -i\hbar (yP_z - zP_y) \end{aligned}$$

$$[L_z, L_y] = -i\hbar L_x$$

$$\begin{aligned} \text{(III)} [L_x, L_z] &= [yP_z - zP_y, xP_y - yP_x] \\ &= [yP_z, xP_y] - [zP_y, xP_y] - [yP_z, yP_x] + [zP_y, yP_x] \\ &= [yP_z, xP_y] - 0 - 0 + [zP_y, yP_x] \\ &= [yP_z, xP_y] + [zP_y, yP_x] \\ &= xP_z [y, P_y] + zP_x [P_y, y] \\ \therefore [L_x, L_z] &= xP_z (i\hbar) + zP_x (-i\hbar) \\ &= i\hbar (xP_z - zP_x) \\ &= -i\hbar (zP_x - xP_z) \end{aligned}$$

$$[L_x, L_z] = -i\hbar L_y$$

These are the fundamental commutation relation for angular momentum.

$L_x, L_y$  and  $L_z$  are incompatible observables.

According to the generalized uncertainty principle

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left( \frac{1}{2i} \langle [L_x, L_y] \rangle \right)^2$$

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left( \frac{1}{2i} \langle i\hbar L_z \rangle \right)^2$$

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left( \frac{\hbar}{2} \langle L_z \rangle \right)^2$$

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \frac{\hbar}{2} |\langle L_z \rangle|$$

The square of the total angular momentum is,

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$\begin{aligned} (1) \quad [L^2, L_x] &= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\ &= 0 + L_y [L_y, L_x] + [L_y, L_x] L_y + L_z [L_z, L_x] + [L_z, L_x] L_z \\ &= L_y (-i\hbar L_z) + (-i\hbar L_z) L_y + L_z (i\hbar L_y) + (i\hbar L_y) L_z \\ &= -i\hbar L_y L_z - i\hbar L_y L_z + i\hbar L_y L_z + i\hbar L_y L_z \end{aligned}$$

$$[L^2, L_x] = 0$$

Similarly,  $[L^2, L_y] = 0$

$$[L^2, L_z] = 0$$

$$\therefore [L^2, \vec{L}] = 0$$

$\therefore L^2$  is compatible with each component of  $\vec{L}$ . To find eigenstates of  $L^2$  and  $L_z$ .

$$L^2 f = Af \quad \text{and} \quad L_z f = Mf$$

Now, Using “ladder operator” technique.

Let,

$$L_{\pm} = L_x \pm iL_y$$

Now

$$\begin{aligned} [L_z, L_{\pm}] &= [L_z, L_x] \pm i[L_z, L_y] \\ &= (i\hbar L_y) \pm i(-i\hbar L_x) \\ &= \pm i\hbar L_y \pm \hbar L_x \\ &= \hbar [\pm L_x + iL_y] \\ &= \pm \hbar [L_x \pm iL_y] \end{aligned}$$

*Continued with...Page 5 Onwards.... It's So Gooooood!!!, Buy it now...!*