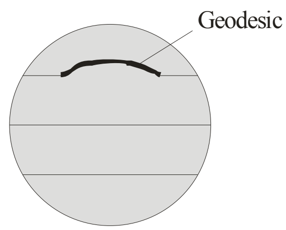
# VOLUME-07 Part B and C

## CONTENTS

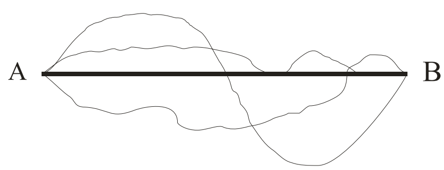
## II. Classical Mechanics

### II.6.1 Hamilton's Variational Principle

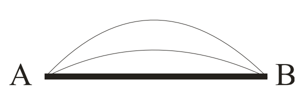
Consider the problem of finding the shortest distance between two points on the surface of a sphere. This is a problem familiar to air travellers, since planes generally do not fly along parallels of latitude even between cities with the same latitude (unless they are taking advantage of particular patterns). The shortest distance between two points is a geodesic, represented by the path can be found qualitatively if one has a model globe handy. By placing a string on the surface of the globe, and then pulling the string taut at the same time as it is forced to pass over the cities of interest, one can see where the geodesic path lies.



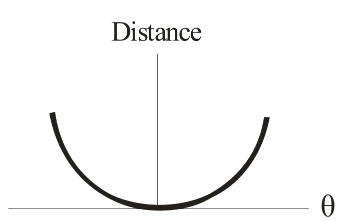

The same things applies to a planar surface: one can imagine many paths between points A and B, but only one represents the minimum distance:




To represent this operation mathematically, imagine some parameter space where the distance is plotted against some variable representing the curve: *e.g.*, suppose all the allowed curves are sections of circles, and we plot distance against angle:




θ is the angle between the curve and the straight line at the origin distance.



The minimal distance corresponds to the value of θ where the derivative of the distance with respect to vanishes:

$$d[distance] / d\theta = 0$$

To lowest order, this is equivalent to saying that the variation of the distance near the true path vanishes

$$\delta[distance] = 0$$

Hamilton proposed in 1834 that this approach could be applied to mechanics, and he rederived a set of equations proposed by Lagrange 50 years earlier in 1788. In Hamilton's approach, it is an integral of the quantity $L = K - V$ that assumes an extremum at the correct path, where

$K$ = Kinetic energy

$V$ = Potential energy

$L$ = Lagrangian.

Mathematically, Hamilton's statement of mechanics is

$$\delta\int_{t_1}^{t_2} Ldt = 0 \qquad \ldots 1$$

The integral is referred to as the action, and the end points of the integral, $t_1$ and $t_2$ are fixed (like the string on the globe).

Example (from Fowles and Cassiday)

As an example of how the extremal principle works, consider the motion of an object falling under gravity. The kinetic and potential energies are

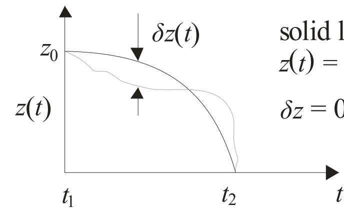
$K = mv^2/2$

$V = mgz$,

where $z$ is the distance in the vertical direction. We do not assume $F = ma$, but rather solve:

$$\delta\int_{t_1}^{t_2}\left[\frac{m}{2}\left(\frac{dz}{dt}\right)^2 - mgz\right]dt = 0 \qquad \ldots 2$$

What does the variation $\delta z$ mean? We know (after the fact) that the correct path is $z = z0 - gt^2/2$. A variation $\delta z(t)$ might then look like




The variation in $v^2 = (dz/dt)^2$ is obtained via the chain rule

$$\delta\left(v^2\right) = 2v\delta v\,(v2)\ = 2vd(\delta z)/dt$$

Hence,

$$\int_{t_1}^{t_2}\left[\frac{m}{2}2v\frac{d(\delta z)}{dt} - mg\delta z\right]dt = 0$$

or

$$\int_{t_1}^{t_2}\left[v\frac{d(\delta z)}{dt} - g\delta z\right]dt = 0 \qquad \ldots 3$$

Now, we have to work at the first term a bit. We know that

$$d(v\ \delta z)/dt = (dv/dt)\ \delta z + v\ d(\delta z)/dt$$

so

$$\int\left[d(v\delta z)/dt\right].dt = \int\left[(dv/dt)\delta z\right].dt + \int\left[vd(\delta z)/dt\right].dt$$

from which we obtain, using $a = dv/dt$,

$$(vdz)\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\left[\frac{(dv)}{dt}\delta z\right]dt + \int_{t_1}^{t_2}\left[v\frac{d(\delta z)}{dt}\right]dt$$

But the left hand side vanishes, because the end-points are fixed, so

$$\int_{t_1}^{t_2}\left[\frac{(dv)}{dt}\delta z\right]dt = -\int_{t_1}^{t_2}\left[v\frac{d(\delta z)}{dt}\right]dt \qquad \ldots 4$$

Substituting (4) into (3) gives

$$\int\left[(a-g)\delta z\right].dt = 0 \qquad \ldots 5$$

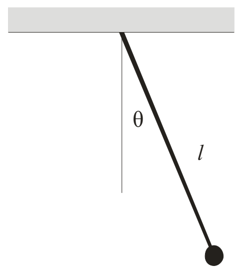
Now, the only way that Equation (5) can vanish for arbitrary variations in $\delta z$ is for $-\,a - g$ to vanish everywhere. That is,

$$a = -\,g$$

which is what we expect from Newton's laws (by a much more obvious calculation). Clearly, we don't want to go through this agony every time we solve a problem, so in the next two lectures we derive a set of equations based upon the variational approach.

**Generalized coordinates**

We have dealt frequently with Cartesian coordinates in this course, in part because their orthogonality prevents cross terms in expressions like **r** . **r**. But we know that there are situations like circular motion wherein it is much more convenient to use angular coordinates, at the expense of introducing explicit weights (like $rd\theta$ ) or cross terms. We introduce a set of generalized coordinates $q_i$ and their time derivatives to describe the motion of particles. By definition, the number of generalized coordinates is equal to the number of degrees of freedom. An example is the simple pendulum




Although the motion of the pendulum lies in a vertical plane, requiring two Cartesian coordinates (say $x$ and $y$), in fact there is only one degree of freedom because $x$ and $y$ are subject to the constraint $x^2 + y^2 = l^2$. Thus, it is better to use a coordinate like θ or arc length $r$ θ, with only one equation of motion, than it is to use $x$, $y$ with two equations of motion that incorporate a constraint.

***Transformations***

Lastly, we just restate that we can transform back and forth between coordinate systems by means of the usual equations:

$$x = x(q_i) \;\; i = 1, n$$

$$dx / dt = \sum_{i=1} n(\partial x / \partial q_i)(dq_i / dt)$$

or

$$\dot{x} = \sum_{i=1} n(\partial x / \partial q_i)\dot{q}_i$$

etc., where $n$ is the number of degrees of freedom.

*Continued with...Page 5 Onwards.... It's So Goooood!!!, Buy it now...!*