VOLUME-04 Part B and C

CONTENTS

I. Mathematical Methods of Physics

I.4. Elements of complex analysis and Laurent series-poles, residues and evaluation of integrals

Elements of Complex Variables

1. DEFINIATIONS

Complex numbers. An ordered pair of real numbers such as (*x, y*) is termed as a complex number. If we write

$$z = (x,y) \text{ or } x + iy, \text{where } i = \sqrt{-1}, \text{then}$$

is called the *real part* and *y the imaginary part* of the complex number *z* and denoted by

$$x = R_z \text{ or } R(z) \text{or } Re(z)$$
$$y - I_z \text{ or } I(z) \text{or } Im(z)$$

Equity of complex numbers. Two complex numbers (*x, y*) and (*x' , y'*) are equal iff *x = x'* and *y = y'*.

Modulus of a complex number. If *z = x + iy* be a complex number then its modulus (or module) is denoted by | *z* | and given by

$$|z| = |x + iy| = +\sqrt{x^2 + y^2}$$

Evidently $|z| = 0 \text{ iff } x = 0, \ y = 0.$

2. OPERATION OF FUNDAMENTAL LAWS OF ALGEBRA ON COMPLEX NUMBERS

Taking three complex numbers $z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_3 = (x_3, y_3)$ we define the following operations:

[1] Addition. The sum of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ (say) is defined as a complex number $z = (z_1, z_2) = (x_1 + x_2, y_1 + y_2)$ such that its real part is the sum of real parts and imaginary part is the sum of imaginary parts of the given numbers.

(*i*) Addition is Commutative. $i.e. z_1 + z_2 = z_2 + z_1$ …1

Since we have $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$$- (x_1 + x_2) + i(y_1 + y_2)$$
$$= (x_1 + x_2, y_1 + y_2)$$
$$= (x_2 + x_1, y_2 + y_1), \text{ all the numbers being real}$$
$$= (x_2 + iy_2) + (x_1 + iy_1)$$
$$= z_2 + z_1$$

(*ii*) Addition is associative. $i.e. z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ …2

Since, we have

$$z_1 + (z_2 + z_3) = x_1 + iy_1 + (x_2 + iy_2 + x_3 + iy_3)$$
$$= (x_1 + iy_1 + x_2 + iy_2) + x_3 + iy_3$$
$$= (z_1 + z_2) + z_3$$

(*iii*) There exist an additive identity *i.e.* **z + 0 = z** …3

Since, we have $\mathbf{z} + \mathbf{0} = (x, y) + (0,0)$

$$= (x + 0, y + 0)$$
$$= (x, y) = \mathbf{z}$$

(iv) **There exists an additive identity** *i.e.* **z + (- z) = 0** …4

Since, $z + (-z) = (x, y) + (-x, -y)$
$= (x - x, y - y)$
$= (0,0)$
$= 0$

Note. If $z = (x, y)$ then $-z = (-x, -y)$ is called as additive inverse of z.

[2] Subtraction. If $z_1 = (x_1, y_1)$ then $-z = (-x_1, -y_1)$ etc.

$\therefore\ z_1 - z_2 = (x_1, y_1) + (-x_2, -y_2) = x_1 + iy_1 - x_2 - iy_2$
$= (x_1 - x_2, y_1 - y_2)$ …5

[3] Multiplication. We have $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$
$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

i.e. $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$ …6

(i) **Multiplication is commutative.** i.e. $z_1z_2 = z_2z_1$ …7

Since, $z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$ by (6)
$- (x_2x_1 - y_2y_1, y_2x_1 + y_1x_2)$
$= (x_2 + iy_2)(x_1 + iy_1)$
$= z_2z_1$

(ii) **Multiplication is associative.** $z_1(z_2z_3) = (z_1z_2)z_3 = z_1z_2z_3$ …8

Since $z_1(z_2z_3) = (x_2, x_3)[x_2x_3 - y_2y_3, x_2y_3 + x_3y_2]$ by (6)
$= [x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + x_3y_2),$
$x_1(x_2y_3 + x_3y_2) + y_1(x_2x_3 - y_2y_3)]$ by (6)
$= [(x_1x_2 - y_1y_2)x_3 - (x_1y_2 - x_2y_1)y_3, (x_1y_2 + x_2y_1)x_3$
$+ (x_1x_2 - y_1y_2)y_3]$ (on rearranging)
$= [(x_1, y_1)(x_2, y_2)]\ (x_3, y_3)$
$= (z_1z_2)z_3$

(iii) **Multiplication is distributive.** *i.e.* $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$ …9

Since $(z_1 + z_2)z_3 = (x_1 + x_2, y_1 + y_2)]\ (x_3, y_3)$
$= [(x_1 + x_2)x_3 - (y_1 + y_2)y_3, (x_1 + x_2)y_3$
$+ (y_1 + y_2)x_3]$ by (6)
$= [(x_1x_3 - y_1y_3) + (x_2x_3 - y_2y_3); (x_1y_3 + x_3y_1)$
$+ (x_2y_3 + x_3y_2)]$ (on arranging)
$= (x_1x_3 - y_1y_3, x_1y_3 + x_3y_1) + (x_2x_3 - y_2y_3, x_2y_3 + x_3y_2)$
$= (x_1, y_1)(x_3, y_3) + (x_2, y_2)(x_3, y_3)$
$= z_1z_3 + z_2z_3$

(iv) **There exists a multiplicative identity** ***i.e.*** **z . 1 = z** …10

Where **1** = (1, 0) is the multiplicative identity known as *unity* for the system of complex numbers.

We have $z\ .\ 1 = (x, y)(1, 0)$
$= (x, y)$
$= z$

(v) **There exists a multiplicative inverse** *i.e.* $zz^{-1} = 1$ …11

If $z = (x, y)$, then $z^{-1} = (x, y)^{-1}$ so that we have to show that

$$(x, y)(x, y)^{-1} = (1,0)$$

Assuming $(x, y)^{-1} - (x', y')$, this becomes

$$(x, y)(x', y') = (1,0)$$

i.e. $(xx' - yy', xy' + yx') = (1,0)$

which gives $xx' - yy' = 1$ (on equating real and imaginary parts)

$$xy' + yx' = 0$$

Solving these equations we get

$$x' - \frac{x}{x^2+y^2}\ y' - \frac{-y}{x^2+y^2}$$ provided $x^2 + y^2 \neq 0$

Hence the complex number (x, y) has a unique multiplicative inverse $\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ which is also a complex number such that (x, y) $\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = (1,0)$

[4] **Division.** Consider an equation $z_1 z_2 = z'$

Where $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ and $z' = (x', y')$

Now $z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) = z' = (x', y')$

which gives $x_1 x_2 - y_1 y_2 = x'$

$$x_1 y_2 + x_2 y_1 = y'$$

Solving $x_2 = \frac{y_1 y' - x_1 x}{x_1^2 + y_1^2}, y_2 = \frac{y_1 y' - x' y_1}{x_1^2 + y_1^2}$ …12

Provided $x_1^2 + y_1^2 \neq 0$ *i.e.* $|z_1| \neq 0$

Thus we have a unique solution and $z_2 = \frac{z'}{z_1}$ is the quotient.

[5] Conjugate complex numbers. If $z = x + iy$, then $x - iy$ is said to be the conjugate of complex number z and denoted by $\bar{z}$

Evidently $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$ …13

$\overline{z_1\, z_2} = \bar{z}_1\, \bar{z}_2$ …14

$\overline{zz} - (x + iy)(x - iy) - x^2 + x^2 - |z|^2$ …15

$z + \bar{z} = 2x = 2\, R_z$ *or* $2R(z)$ …16

$z - \bar{z} = i2y = i2\, I_z$ *or* $2\, iI(z)$ …17

3. GRAPHICAL REPRESENTATION (ARGAND DIAGRAM)

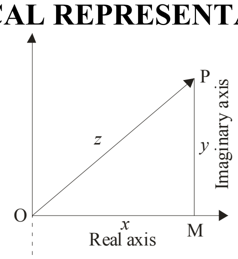

Fig.1

Consider a point P in xy − plane. Let an ordered pair of values of a and y correspond to the co-ordinates of the point P. Then a complex number *z* may be made to correspond to the point P, where

$$z = x + iy.$$

Here *z* is called the complex the complex co-ordinate of the point P.

In the adjoining figure, the *x-axis* is called the real *axis* or *axis* of *reals* and *y*-axis is called the *imaginary axis* or the *axis of imaginaries.*

Here $|z| = |x + iy| = \sqrt{(x^2 + y^2)}$ is the measure of length *OP*.

It (r, θ) be the polar co-ordinates of the point P, the polar form of the complex number *z* is

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Here the number *r* (being taken + ive) is called the **modulus** or **absolute value** of the complex number *z* and θ is called the **angle** or **argument** of *z* and usually written as arg *z*, *i.e.,*

$|z| = r$ and arg z = θ.

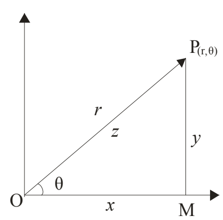

Fig. 2

Now the co-ordinates of a point P' which is conjugate of *z* are $\bar{z} = (x, -y)$ or $(r, -\theta)$ in polars.

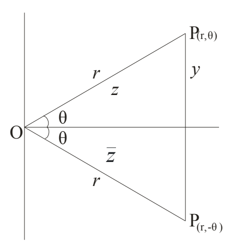

Fig. 3

Since $\bar{z} = r\{\cos(-\theta) + i\sin(-\theta)\}$, geometrically the points *P* and *P'* represent *z* and $\bar{z}$ respectively and their situations are symmetrical about the axis of relas, *i.e x*-axis. The conjugate of *z* is called the **reflection** or **image** of *z* in the real axis.

Note 1. The plane whose points are represented by complex numbers is known as **Argand** plane or **Argand** diagram or **Complex plane** or **Gaussian plane.**

Note 2. The complex number *z* representing the point (*x*, *y*) is sometimes called as *Affix* of the point (*x*, *y*).

Note 3. The sum, difference, product and quotient of complex numbers can be geometrically represented on the Argand plane as follow: