VOLUME-03 Part B and C

CONTENTS

I. Mathematical Methods of Physics

I.3.Linear Differential Equations and Special functions

1. Solution in Simple cases of ordinary differential equations of second order

Generally an ordinary equation of second order is of the from

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

Where P, Q, X, are function of x only.

I. Linear Equations with Constant Coefficients.

The general solution is found by usual methods of finding the complementary function and particular integral of the equation. Although the student is presumed to have a sound knowledge of the methods to be employed for finding the complementary function and particular integral, but still then, we summarize them as below:

Let there be a differential equation of the type

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2y = X$$

Where P_1, P_2 are constants and X is a function of x.

In terms of the D operator, it may be written

$(D^2 + p_1D + p_2)y = X$ where D stand for $\frac{d}{dx}$ i.e.; $D \equiv \frac{d}{dx}$

Or we may write thus, $f(D)y = X$

To find the complementary function (C. F). The X is removed and replaced by zero. Then an auxiliary equation is written either by putting $y = e^{mx}$ whence, we get

$$m^2 + p_1m + p_2 = 0$$

Or simply writing $D^2 + p_1D + p_2 = 0 \; i.e., f(D) = 0$

In either case, we get the roots of the quadratic.

Case I. If the roots are of the type $m_1 \; and \; m_2$ (real and district) C. F, is

$$C_1e^{m_1x} + C_2e^{m_2x}$$

Case II. $m_1 = m_2$ i.e., both the roots are real and equal, C. F, is

$$(C_1 + C_2x)e^{m_1x}$$

Case III. If the roots are imaginary i.e., of the type $\alpha \pm i\beta$ C. F, is

$$e^{\alpha x}[C_1\cos\beta x + C_2\sin\beta x] \;\; or \;\; C_1e^{\alpha x}\cos(\beta x + C_2).$$

Case IV. If the roots are of the type $\alpha \pm \sqrt{\beta}$, C.F, is

$$C_1e^{\alpha x}\cosh\left(\sqrt{\beta}x + C_2\right)$$

Note. The number of the arbitrary constants will be the same as the order of the equation.

To find the Particular Integral (P.I.)

We have $P.I. = \frac{X}{f(D)}$ which for $f(D) = D - \alpha \Rightarrow \frac{X}{D-\alpha} = e^{\alpha x}\int Xe^{-\alpha x}dx.$

Case I. If $X = e^{ax}$ where α is any constant.

$$P.I. = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)} if \; f(a) \neq 0.$$

Case II. If $X = x^m$, where m is a positive integer

$$P.I. = \frac{x^m}{f(D)} = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ binomially upto m^th^ power of D and then operate x^m on every term.

Case III. If $X = \sin ax \quad or \quad \cos ax$

$$P.I. = \frac{\sin ax \; or \; \cos ax}{f(D^2)} = \frac{\sin ax \; or \; \cos ax}{f(-a^2)}$$ provided $f(-a^2) \neq 0$.

In case $f(-a^2) = 0, \frac{\sin ax}{f(D^2)} = \text{Imaginary part of } \frac{e^{iax}}{f(D^2)}$

And $\frac{\cos ax}{f(D^2)} = \text{Real part of } \frac{e^{iax}}{f(D^2)}$, which is case I.

Case IV. $X = e^{ax}$, where V is any function of x, then

$$P.I. = \frac{x^m}{f(D)} = e^{ax}.\frac{1}{f(D+a)}.V.$$

Case V. If $X = x.V$, where V is any function of x, then

$$P.I. = \frac{xV}{f(D)} = x\frac{1}{f(D)}.V - \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V$$

$$= \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V$$

Hence general solution = C. F. + P. I.

Problem 1. $Solve \; \frac{d^2y}{dx^2} - y = x \; \sin x + (1 + x^2)e^x$

$$(D^2 - 1)y = x \; \sin x + (1 + x^2)e^x$$

Now its complementary function and particular integral may be found thus:

For complementary function, the auxiliary equation is

$$D^2 - 1 = 0 \;\; or \;\; D = \pm 1$$

C.F. is $C_1e^x + C_2e^{-x}$

Particular integral $-\frac{x\sin x}{D^2-1}+\frac{(1+x^2)e^x}{D^2-1}$

$$= \text{imaginary part in } \frac{xe^{ix}}{D^2-1}+e^x\frac{1}{(D+1)^2-1}(1+x^2)$$

$$= \text{imaginary part in } e^{ix}\frac{1}{(D+i)^2-1}.x+e^x.\frac{1}{D^2+2D}(1+x^2)$$

$$= \text{imaginary part in } e^{ix}\frac{1}{D^2+2iD-2}\ x+\frac{e^x.1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$$

$$= \text{imaginary part in } \frac{e^{ix}}{-2}\left[1-\frac{2iD+D^2}{2}\right]^{-1}x+\frac{e^x}{2}.D^{-1}\left\{1+\frac{D}{2}\right\}^{-1}.(1+x^2)$$

$$= \text{imaginary part in } \frac{e^{ix}}{-2}\left[1+\frac{2iD+D^2}{2}+\cdots\right]x+\frac{e^x}{2}D^{-1}\left[1-\frac{D}{2}+\frac{D^2}{4}-\cdots\right](1+x^2)$$

$$= \text{imaginary part in } \frac{e^{ix}}{-2}[x+i]+\frac{e^x}{2}D^{-1}\left[1+x^2-x+\frac{1}{2}\right]$$

$$= \text{imaginary part in } \left(\frac{\cos x+i\sin x}{-2}\right)(x+i)+\frac{e^x}{2}\int\left(x^2-x+\frac{3}{2}\right)dx$$

$$= -\frac{1}{2}(\sin x+\cos x)+\frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3}{2}x\right]$$

Hence the general solution is

$$y = C_1e^x+C_2e^{-x}-\frac{1}{2}(x\sin x+\cos x)+\frac{xe^x}{12}(2x^2-3x+9)$$

II. Linear Equations with Variable Coefficients (Homogeneous Linear Equations).

Consider, $P_0x^2\frac{d^2y}{dx^2}+P_1x\frac{dy}{dx}+P_2y=X.$

Put $x=e^x$, i.e., $z=\log x$; $\therefore \frac{dz}{dx}=\frac{1}{x}.$

Then $\frac{dy}{dx}=\frac{dy}{dz}.\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz},$

$$\frac{d^2y}{dx^2}=\frac{1}{x}.\frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right)=\frac{1}{x^2}\frac{d^2y}{dz^2}-\frac{1}{x^3}\frac{dx}{dz}\frac{dy}{dz}$$

$$=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

If we put $\frac{d}{dz}\equiv D$, we have

$$x\frac{dy}{dx}=Dy, x^2\frac{d^2y}{dx^2}=D\ (D-1)\ y,\ldots.x^n\frac{d^ny}{dx^n}=D(D-1)\ldots(D-n+1)$$

Now substituting these values in (1), we get

$$P_0D\ (D-1)\ y+P_1Dy+P_2y=X,$$

Which may be solved by the method employed in I.

Problem 2. *Solve* $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=2\log x.$

Put $x = e^z$ and denote $\frac{d}{dz}$ by D; we have

$$D(D-1)\,y - Dy + y = 2z \;\; or \; (D^2 - 2D + 1)y = 2z.$$

Auxiliary equation is

$$D^2 - 2D + 1 = 0$$

Or $$(D-1)^2 = 0; \quad \therefore D = 1 \;(repeated\ twice).$$

C.F. $= (c_1 + c_2 z)e^z$.

P.I.$= \frac{2z}{D^2 - 2D + 1} = 2[1 - (2D - D^2)]^{-1}.z$

$= 2(1 + 2\,D \ldots)z = 2\,(z + 2) = 2z + 4.$

$\therefore$ General solution is $y = (c_1 + c_2 z)e^z + 2z + 4$

$$= (c_1 + c_2 \log x)x + 2\log x + 4.$$

Note. Any equation of the type

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1 (a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(a+bx)\frac{dy}{dx} + P_n y = F(x)$$

Can be reduced to the homogeneous linear form by putting $z = ax + b$ or this can be solved by putting $ax + b = e^z$ as above.

III. Exact Differential Equations and Equations of other Special types.

The equations of the type

$$P_0 \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$$

Where P_0, P_1 *and* P_2 are the functions of x, is said to be exact if

$$P_2 - P_1 + P_0 = 0$$

Or in general an equation of order n (say),

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n y = Q(x)$$

Is exact if $$P_n - P_{n-1} + \cdots + (-1)^n P_0(n) = 0.$$

Where P', P'', … $P^{(n)}$ are the successive derivatives of P.

In case the equation is exact, its integral is

$$P_0 y_{n-1} + (P_1 - P'_0)y_{n-2} + (P_2 - P'_1 + P_0'')y_{n-2} + \cdots$$
$$\cdots + [P_{n-1} - P'_{n-2} + \cdots + (-1)^n P_0^{n-1}]y = \int Q(x) + C,$$

Where y_n stands for $\frac{d^n y}{dx^n}$ etc.