VOLUME-02 Part B and C

CONTENTS

I. **Mathematical Methods of Physics**

2.1 Linear Algebra, Matrices	1
2.2 Rank of a Matrix	78
2.3 Cayley-Hamilton Theorem	109
2.4 Sub-Spaces and Null Spaces	112

I. Mathematical Methods of Physics

I.2 Linear Algebra, Matrices

1. Definitions and Notations

A set of *mn* numbers, real or complex, arranged in a rectangular array of *m* rows and *n* columns such as

$$\begin{bmatrix} a_{11}, a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21}, a_{22} \ldots\ldots\ldots\ldots a_{2n} \\ \ldots, \ldots \quad \ldots\ldots\ldots \quad \ldots \\ a_{m1}, a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}$$

is called a matrix of order $m \times n$.

In other words a scheme of detached coefficients a_{ij} arranged in *m* rows and *n* columns is called a matrix of order *m* by *n* or an $m \times n$ matrix or a matrix of type $m \times n$.

In case $m = n$, the rectangular array becomes a square and so the matrix having number of rows and number of columns equal is called a Square Matrix of order *n*. Any matrix obtained by deleting any number of rows and any number of columns from a given matrix is said to be a Sub-Matrix of the given matrix.

The *mn* numbers $a_{ij}, (i = 1,2,\ldots\ldots m;\ j = 1,2,\ldots\ldots n, i \neq j)$ constituting the $m \times n$ matrix are called its elements or constituents. The elements $a_{ij}\,(i = j)$ of a square matrix A are called its diagonal elements and their sum as trace of A denoted by tr. $A = \sum_{i=1}^{n} a_{ij}$

A matrix is usually denoted by capital letters like A (in Clarendons type) or $\lfloor a_{ij} \rfloor$, where a_{ij} represents the $(i, j)^{\text{th}}$ element i.e., the element in the i^{th} row and j^{th} column of the matrix.

Thus an $m \times n$ matrix may be expressed as

$$A = \lfloor a_{ij} \rfloor = \begin{bmatrix} a_{11}, a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21}, a_{22} \ldots\ldots\ldots\ldots a_{2n} \\ \ldots, \ldots \quad \ldots\ldots\ldots \quad \ldots \\ a_{m1}, a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix} \begin{matrix} \text{where } 1 \leq i \leq m \\ \text{and } 1 \leq j \leq n \\ \text{but } i \neq j \end{matrix}$$

We have so far used only a pair of brackets i.e.[ ] to denote a matrix, but a pair of parentheses i.e. ( ) and double bars i.e. || ||, are also sometimes used to indicate a matrix.

A matrix having all of its elements zero is said to be a Null Matrix and denoted by 0 e.g.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{or} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A square matrix of order *n* having all its diagonal elements unity and zero elements everywhere else is called a unit matrix or an identity matrix and denoted by I_n. Thus,

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

It is possible that a matrix may have only a single row or a single column such as

$$\begin{bmatrix} a_1, & a_2, & \dots & a_p \end{bmatrix} \text{and} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_q \end{bmatrix}$$

the first one being a matrix of order $1 \times p$ is called a row matrix and the second one being a matrix of order $q \times 1$ is called a column matrix. A single element constitutes a matrix of order 1×1. In relation to matrices, the number are usually known as scalars; for they behave as operators exactly like ordinary numbers as multipliers and hence are called scalars.

ILLUSTRATIVE EXAMPLES

1. $\begin{bmatrix} 2 & 3 & -1 \\ 4 & -5 & 6 \end{bmatrix}$ is a matrix of order 2×3.

2. $\begin{bmatrix} 2 & -3 & 4 \\ 5 & 6 & -2 \\ 1 & 0 & 4 \end{bmatrix}$ is a square matrix of order 3.

3. $\begin{bmatrix} 2 & 3 & -1 \\ 4 & -5 & 6 \end{bmatrix}$ is a sub-matrix of the matrix $\begin{bmatrix} 2 & 0 & 3 & -1 & 2 \\ 4 & 5 & 6 & 7 & 8 \\ 9 & 4 & 2 & 1 & 5 \end{bmatrix}$

4. *4, 0, 6* are the diagonal elements of the matrix

$$A = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 6 & 3 \\ 2 & -5 & 6 \end{bmatrix}$$ whose elements are *4, 5, 6, 2, 0, 3, 2 -5,6 i.e.*

if $\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 0 & 3 \\ 2 & -5 & 6 \end{bmatrix}$ then $a_{11} = 4, a_{12} = 5, a_{13} = 6,$ etc.

also trace of A i.e. tr A = 4 + 0 + 6 = 10.

5. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix.

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.
7. $\begin{bmatrix} 2 & 3 - 5 \end{bmatrix}$ is a 1×3 row matrix.
8. $\begin{bmatrix} 3 \\ 1 \\ 5 \\ 7 \end{bmatrix}$ is 4×1 column matrix.

2. Equality of Matrices

Two matrices A and B defined as

$$A = \lfloor a_{ij} \rfloor \text{ and } B = \lfloor b_{ij} \rfloor$$

are said to be equal if both are of the same order $m \times n$ i.e. A has the same number of rows and columns as B and each element a_{ij} of A is equal to the corresponding element b_{ij} of B i.e. $a_{ij} = b_{ij}$ for each pair of subscripts *i* and *j*.

Hence for equality, the two matrices must be identical in every respect or broadly speaking, the two matrices are equal if and only if one is a duplicate of the other.

ILLUSTRATIVE EXAMPLES

1. If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ and B = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

then A = B if and only if

$$a_{11} = b_{11}, a_{12} = b_{12}, a_{21} = b_{21}, a_{22} = b_{22}\ a_{31} = b_{31}, a_{32} = b_{32}$$

2. The matrices $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 5 \\ 3 - 4 & 0 \end{bmatrix}$ being of different order are not comparable for equality.

3. The matrices $\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 5 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 6 & 1 \\ 5 & 4 \end{bmatrix}$ are comparable but not equal as the element of the 2^{nd} row and 1^{st} column of the first matrix is not equal to the corresponding element of the second matrix.

4. The matrices $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 9 & 7 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 9 & 7 \end{bmatrix}$ are equal.

5. The matrices $\begin{bmatrix} 4 & 5 & 27 \\ 6 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2^2 & 5 & 3^3 \\ 3.2 & 0 & \sqrt{9} \end{bmatrix}$ are equal.

COROLLARY: **Equivalence Relation on Matrices**

If there are three matrices A, B, C such that they satisfy the following three properties.

(1) Reflexivity	A = A

(2) Symmetry	A = B implies that B = A

(3) Transitivity	A = B and B = C imply that A = C

Then the equality of matrices is said to form an equivalence relation.

3. Addition of Matrices

Two matrices A $=[a_{ij}]$ and B $=[b_{ij}]$ are said to be conformable for addition if they are of the same order i.e. they have the same number of rows and the same number of columns. The sum of the two matrices A and B is then defined as the matrix each of whose elements is the sum of corresponding elements of A and B i.e.

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

For example,

$$\text{if } A = \begin{bmatrix} a_{11}, a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21}, a_{22} \ldots\ldots\ldots\ldots a_{2n} \\ \ldots, \ldots \quad \ldots\ldots\ldots \quad \ldots \\ a_{m1}, a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11}, b_{12} \ldots\ldots\ldots\ldots b_{1n} \\ b_{21}, b_{22} \ldots\ldots\ldots\ldots b_{2n} \\ \ldots, \ldots \quad \ldots\ldots\ldots \quad \ldots \\ b_{m1}, b_{m2} \ldots\ldots\ldots\ldots b_{mn} \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} a_{11} + b_{11}, & a_{12} + b_{12}, & \ldots\ldots & a_{1n} + b_{1n}, \\ a_{21} + b_{21}, & a_{22} + b_{22}, & \ldots\ldots & a_{2n} + b_{2n}, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ a_{m1} + b_{m1}, & a_{m2} + b_{m2}, & \ldots\ldots & a_{mn} + b_{mn}, \end{bmatrix}$$

As another example if

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -4 & 1 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 0 \\ 3 & -4 & -5 \end{bmatrix}$$

$$\text{then } A + B = \begin{bmatrix} 2-1 & 0+2 & 3+0 \\ -4+3 & 1-4 & 5-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 0 \end{bmatrix}$$

COROLLARY 1. $tr(A+B) = trA + trB$

e.g. if $A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

then $tr(A+B) = 1 + 4 = 5 = 2 + 3 = trA + trB$

Continued with...Page 5 Onwards.... It's So Goooood!!!, Buy it now...!